Analytical Pricing of CDOs in a Multi-factor Setting by a Moment Matching Approach

Antonio Castagna

Fabio Mercurio

Paola Mosconi

\(^1\)Iason Ltd.

\(^2\)Bloomberg LP.

\(^3\)Banca IMI

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2. The Credit Portfolio Model
   - Stylized Facts
   - Quantile Expansion for large $q$
   - The Moment Matching Method
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   - CDOs: Stylized Facts
   - Gaussian Copula Approximation
   - Correlation
4. Approximating Distributions
5. Implementation and Numerical Analysis
6. Conclusions
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We propose a credit portfolio model which allows to treat in a unified framework Credit VaR and the risk of structured products, i.e. the Economic Capital required to face unexpected credit losses, and the risk transferred out of the balance sheet via the securitization activity.

Analytical approximation of value at risk [1] + Analytical pricing of Collateralized Debt Obligations (CDOs)
The starting point is the so called “Asymptotic Single Risk Factor” (ASRF) paradigm originally introduced by Vasicek in 1991 [2], which stands at the basis of the Basel 2 Capital Accord, Pillar I, and whose hypothesis has been used to price CDOs [3].

ASRF is based on the assumption of a large and homogeneous portfolio of loans, where only a single driver of systematic risk is present and where the dependence structure among different obligors is described the the so-called Gaussian Copula.

We extend the ASRF model, in order to include a richer structure of dependencies (multi-factor, inhomogeneities, contagion) still retaining the analytical tractability of the model, which is generally lost when complexity is added.
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We consider a portfolio consisting of $M$ distinct loans.

- Each debtor is characterized by a probability of default $PD_i$ and is associated a stochastic variable which describes the value of its assets:

$$X_i = r_i Y_i + \sqrt{1 - r_i^2} \xi_i,$$

(Y_i multi-factor systematic risk driver, $\xi_i$ idiosyncratic term, $r_i$ factor loadings).

- Default occurs if $X_i$ falls below a threshold equal to $N^{-1}(PD_i)$.

- Each loan is characterized by a loss given default $LGD_i$ and an exposure $w_i$. The portfolio loss is given by the sum of individual losses:

$$L = \sum_{i=1}^{M} w_i L_i = \sum_{i=1}^{M} w_i \mathbb{I}\{X_i \leq N^{-1}(PD_i)\} LGD_i.$$
The Credit Portfolio Model (II)

The portfolio loss $L$ is analytically tractable and allows to extract important information about:

- the quantile of $L$ at a large confidence level $q$ (through a second order Taylor expansion [5]). This quantity is fundamental in the calculation of the value at risk ($VaR_q$) entering the economic capital, where $q = 99.9\%$;

- the structuring of collateralized debt obligations (CDOs), for lower values of $q$ (typically, up to $q = 22\%$). In particular, we will show how to price analytically CDO’s tranches, by means of a moment matching technique.
Quantile Expansion for large $q$: $\text{VaR}_{99.9\%}$

This topic has been dealt with extensively in [1] and references therein. Here, we summarize qualitatively the main results.

Violations of the hypothesis underlying the ASRF model give rise to corrections which are explicitly taken into account by the BCBS [4] under the generic name of concentration risk. They can be classified in the following way:

- **Name concentration**: “imperfect diversification of idiosyncratic risk”, i.e. imperfect granularity in the exposures ($G$)
- **Sector concentration**: “imperfect diversification across systematic components of risk” ($S$)
- **Contagion**: “exposures to independent obligors that exhibit default dependencies, which exceed what one should expect on the basis of their sector affiliations” ($C$)

$$\text{VaR}_{99.9\%} = \text{VaR}^{\text{ASRF}}_{99.9\%} + (\Delta G + c) + (\Delta S + c)$$
The Moment Matching Method (MMM)

This technique stays at the basis of the analytical pricing of CDOs.

Consider a random variable \( X \) whose distribution is unknown, but whose moments \( M_1, M_2, \ldots, M_n \) can be computed. Then, \( X \) can be replaced by a random variable \( Y \) with known distribution and whose first \( n \) moments match those of \( X \).

The portfolio loss distribution \( L \) is unknown but its moments of arbitrary order can be analytically computed. We choose an approximating distribution \( L^* \), whose distributional properties are known, and apply the MMM. Depending on the number of parameters \( n \) which define \( L^* \), we calibrate the first \( n \) exact moments \( M_n^* \) of \( L^* \) to the corresponding moments \( M_n \) of \( L \) and derive the values of the \( n \) parameters.

We propose a two- and four-parameter approximating distribution.
Synthetic CDOs with maturity $T_b$ are contracts obtained by putting together a collection of Credit Default Swaps (CDS) with the same maturity on different names and then tranching the loss associated to this pool, at two detachment points $A$ and $B$ with $0 \leq A < B \leq 1$.

The tranched loss at a generic time $t$ is given by:

$$L_t^{AB} \equiv \frac{1}{B - A} \left[ (L_t - A) \mathbb{I}_{A < L_t \leq B} + (B - A) \mathbb{I}_{L_t > B} \right].$$
CDOs (II)

- Given a set of maturities $T_i = T_1, \ldots, T_b$ and assuming that payments are exchanged at the end of each period, the periodic premium (spread) paid by the protection buyer to the protection seller, in exchange for payments in case of loss in the $A$-$B$ tranche is given by:

$$\text{Spr}_{AB} = \frac{\sum_{i=1}^{b} P(0, T_i) \left( \mathbb{E}[L_i^{AB}] - \mathbb{E}[L_{i-1}^{AB}] \right)}{\sum_{i=1}^{b} P(0, T_i)(T_i - T_{i-1}) \left( 1 - \frac{\mathbb{E}[L_i^{AB}] + \mathbb{E}[L_{i-1}^{AB}]}{2} \right)}.$$

The percentage losses $\mathbb{E}[L_i^{AB}]$ are called Expected Tranche Losses (ETLs) and represent building blocks in the pricing of CDOs:

$$\mathbb{E}[L_i^{AB}] = \frac{\mathbb{E}[(L_i - A)^+] - \mathbb{E}[(L_i - B)^+]}{B - A},$$

($P(0, T_i)$ discount factors.)
The Gaussian Copula Approximation is based on the ASRF model, i.e. it assumes a homogeneous ($PD_i = p$ and $r_i = r$ for each obligor $i$), large portfolio with a single systematic risk factor and no contagion.

Under these assumptions the Expected Tranche Loss (ETL) for tranche $[A, B]$ admits a closed form expression:

$$
\mathbb{E}[L^A_B] = \frac{N_2(N^{-1}(p), -N^{-1}(A), -\sqrt{1 - r^2}) - N_2(N^{-1}(p), -N^{-1}(B), -\sqrt{1 - r^2})}{B - A}
$$

($N(\cdot)$ and $N_2(\cdot, \cdot)$ indicate the normal cumulative and bivariate cumulative distribution functions.)
The GCLHP approach represents a standard tool to extract implied default correlations from market quotes and works as an interpolator in order to quote off-market spreads.

Two types of implied correlations:

- **compound correlation**, obtained by fixing a tranche and inverting the pricing formulae (2) and (4), such that the correlation parameter \( r \) yields the market spread;

- **base correlation**, extracted from the spreads associated to 0-A and the fictive tranches 0-\( B_1 \), 0-\( B_2 \)... In general, it requires a bootstrapping procedure. In our case, spreads of the 0-\( B_1 \), 0-\( B_2 \)... tranches are easily obtained within the model.

Stripping **base correlation** is analogous to stripping implied volatilities of equity options and, being such correlation monotonic in spread, it is always invertible.
Consider the approximating portfolio loss $L^*$ to be distributed according to:

$$L^* \sim \text{Vasicek}(p, r) \quad 0 < p, r < 1.$$ 

- Cumulative distribution function (CDF):

$$F_{p,r}(x) = P(L^* \leq x) = N\left(\frac{\sqrt{1-r}N^{-1}(x) - t}{\sqrt{r}}\right),$$

with $0 \leq x \leq 1$ and $t \equiv N^{-1}(p)$;

- Moments of order $n$ are analytically computed by means of:

$$M_n^* = N_n(t, \ldots, t, \Sigma_{n \times n})$$

where $N_n()$ indicates the $n$-dimensional normal cumulative distribution function and $\Sigma_{n \times n}$ the correlation matrix with entries $\Sigma_{ii} = 1$ and $\Sigma_{ij} = r$;

$$\mathbb{E}[(L^* - x_0)^+] = N_2[N^{-1}(p), -N^{-1}(x_0), -\sqrt{1-r}].$$
We consider a mixture of two Vasicek distributions, Vasicek($p_1$, $r$) and Vasicek($p_2$, $r$).

- **Cumulative distribution function (CDF):**
  \[
  F_{a,p_1,r,p_2}(x) = a \ N \left[ \frac{\sqrt{1-r} N^{-1}(x) - t_1}{\sqrt{r}} \right] + (1-a) \ N \left[ \frac{\sqrt{1-r} N^{-1}(x) - t_2}{\sqrt{r}} \right]
  \]
  where $0 \leq a, r, p_1, p_2 \leq 1$ and $t_i \equiv N^{-1}(p_i)$, $i = 1, 2$.

- **Moments of order $n$ are given by:**
  \[
  M_n^* = a \ N_n(t_1, \ldots, t_1, \Sigma_{n \times n}) + (1-a) \ N_n(t_2, \ldots, t_2, \Sigma_{n \times n})
  \]

- **Expected value:**
  \[
  \mathbb{E}[(L^* - x_0)^+] = a \ N_2[N^{-1}(p_1), -N^{-1}(x_0), -\sqrt{1-r}] + (1-a) \ N_2[N^{-1}(p_2), -N^{-1}(x_0), -\sqrt{1-r}].
  \]
Model Specification

Portfolio:
- $M = 200$ names and $N = 4$ industry-geographic sectors;
- four rating classes. The average one year probability of default under the risk neutral measure is $p_{ave} \approx 3.71\%$ and it is assumed to increase at a constant rate of 2% per year. Loss given default is considered constant through time and its average value is fixed at $LGD_{ave} \approx 40\%$;
- the portfolio is neither homogeneous in its exposures nor concentrated in any particular name or sector. Contagion effects are present.

CDO structure:
- tranches cover losses between:
  - equity tranche: $0\%-3\%$
  - mezzanine tranches: $3\%-6\%, 6\%-9\%, 9\%-12\%$
  - senior tranches: $12\%-22\%, 22\%-100\%$
- maturities $T = 3, 5, 7, 10$ years.
Calibration Results (I)

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3y</td>
<td>0.0157</td>
<td>0.1144</td>
</tr>
<tr>
<td>5y</td>
<td>0.0164</td>
<td>0.1116</td>
</tr>
<tr>
<td>7y</td>
<td>0.0170</td>
<td>0.1100</td>
</tr>
<tr>
<td>10y</td>
<td>0.0181</td>
<td>0.1075</td>
</tr>
</tbody>
</table>

Table: Vasicek calibration.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3y</td>
<td>0.1764</td>
<td>0.0448</td>
<td>0.0095</td>
<td>0.0232</td>
</tr>
<tr>
<td>5y</td>
<td>0.1756</td>
<td>0.0459</td>
<td>0.0101</td>
<td>0.0224</td>
</tr>
<tr>
<td>7y</td>
<td>0.1615</td>
<td>0.0493</td>
<td>0.0108</td>
<td>0.0202</td>
</tr>
<tr>
<td>10y</td>
<td>0.1603</td>
<td>0.0521</td>
<td>0.0116</td>
<td>0.0176</td>
</tr>
</tbody>
</table>

Table: Mixture of Vasicek calibration.
Calibration Results (II)

Comparison between the Vasicek and the mixture of Vasiceks distributions:

<table>
<thead>
<tr>
<th>$k$</th>
<th>Exact $M_k$</th>
<th>Mixture $M^*_k$</th>
<th>$\text{abs err}$</th>
<th>$% \text{ err}$</th>
<th>Vasicek $M^*_k$</th>
<th>$\text{abs err}$</th>
<th>$% \text{ err}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01570</td>
<td>0.01573</td>
<td>2.69E-05</td>
<td>0.17%</td>
<td>0.01570</td>
<td>0</td>
<td>0.00%</td>
</tr>
<tr>
<td>2</td>
<td>0.00048</td>
<td>0.00048</td>
<td>3.19E-06</td>
<td>0.67%</td>
<td>0.00048</td>
<td>3.340E-08</td>
<td>0.01%</td>
</tr>
<tr>
<td>3</td>
<td>2.25E-05</td>
<td>2.25E-05</td>
<td>3.06E-08</td>
<td>0.14%</td>
<td>2.36E-05</td>
<td>1.101E-06</td>
<td>4.89%</td>
</tr>
<tr>
<td>4</td>
<td>1.24E-06</td>
<td>1.29E-06</td>
<td>4.94E-08</td>
<td>3.97%</td>
<td>1.72E-06</td>
<td>4.764E-07</td>
<td>38.32%</td>
</tr>
</tbody>
</table>

Table: Calibration results for $T = 3$ years.

In general the two distributions are practically equivalent as far as lower moments are considered, but a relative error ranging from 30%-40% is always present in the two parameter Vasicek case.
CDO Pricing: Tranches’ Spreads

Table: Spreads obtained for the Vasicek distribution model. The upfront amount* is equal to $U_{0\%3\%} = 0.217$

<table>
<thead>
<tr>
<th>Spread Vasicek</th>
<th>0-3%</th>
<th>3-6%</th>
<th>6-9%</th>
<th>9-12%</th>
<th>12-22%</th>
<th>22-100%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.00%</td>
<td>0.87%</td>
<td>0.15%</td>
<td>0.03%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Table: Spreads obtained for the mixture of Vasicek distributions model. The upfront amount* is equal to $U_{0\%3\%} = 0.1862$.

<table>
<thead>
<tr>
<th>Spread Mixture</th>
<th>0-3%</th>
<th>3-6%</th>
<th>6-9%</th>
<th>9-12%</th>
<th>12-22%</th>
<th>22-100%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.00%</td>
<td>1.19%</td>
<td>0.15%</td>
<td>0.01%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

* For the equity tranche ($A = 0$) it is common practice to quote the upfront amount $U_{0B}$ needed to make the contract fair when a running spread of 500bps is taken as a periodic spread in the premium leg.
A smile behavior in the implied base correlation similar to that observed in the equity option market appears, consistently with the existing literature (see e.g. [6], [7] and [8]). The smile behavior is due to the fatness of the tails in true portfolio loss distribution, which is not captured by a Gaussian Copula LHP approach, but that is implicit, by construction, in the two models we have proposed.
Compound correlation shows behaviors consistent with patterns already encountered in literature (see e.g. Brigo et al. [6]) as a reflection of the difficulty of extracting correlations on mezzanine tranches, where spread is not a monotonic function of correlation. This is related to the existence of multiple solutions to the inversion problem.
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We have considered the credit portfolio model presented in [9] and [1], which describes an extension of the ASRF paradigm including several kinds of dependencies and inhomogeneities.

By means of a **moment matching** technique, we have provided fast and accurate analytical formulas to price CDOs’ tranches. The results obtained are consistent with the existing literature and, in terms of the implied correlation, a smile behavior is found, which can be explained by the fat-tail nature of the true loss distribution.

The model allows also to evaluate analytically, by means of a Taylor expansion, the value at risk of the portfolio, without resorting to computationally intensive methods.

We have proposed a model which is rich and complete, considering all the fundamental risks of a credit portfolio, and which requires little numerical and computational efforts.


